Subspace Identification of Poorly Excited Industrial Systems

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Abstract— **Most of the industrial applications are multiple input multiple output (MIMO) systems, that can be be identified using knowledge of the system's physics or from measured data employing statistical methods. Currently, there is the only class of statistical identification methods capable of handling the issue of vast MIMO systems – subspace identification methods. These methods, however, as all statistical methods, need data of certain quality, i.e. excitation of corresponding order, data corruption, etc. Nevertheless, the combination of statistical methods and physical knowledge of the system could significantly improve system identification. This paper presents a new algorithm which provides remedy to insufficient data quality of certain kind through incorporating of prior information, e.g. known static gain or input-output feedthrough. The presented algorithm naturally extends classical subspace identification algorithms, that is, it adds extra equations into the computation of system matrices. The performance of the algorithm is shown on a case study, where the model is used for an MPC control of a large building heating system.**

Index Terms— **Subspace methods; Identification for control**

I. MOTIVATION

With 39 %, buildings contribute significantly to total energy usage in 2005, as stated by the U. S. Energy Information Administration [1]. This poses strong motivation for creation of advanced and energy saving HVAC (Heating, Ventilating, and Air Conditioning) systems [2]. Significant amount of energy can be saved using predictive control strategies (Project OptiControl[1]) compared to the conventional strategies. Widely used control strategy, weather-compensated control, can lead to poor energy management or reduced thermal comfort even if properly set up, because it utilizes current outside temperatures only. In case of sharp change of weather, there is an improper control action due to the energy accumulation in large buildings, resulting in over- or underheating of the building. Even though HVAC control systems have been improved significantly during recent years, predictive controller described in [3] introduces a different approach to the heating system control design. There is, however, a crucial condition for the successful control, that is, properly identified model of the system. Model identification can be performed by variety of methods, physical modeling or statistical approach among others.

This paper presents subspace identification methods as a tool for identification of MIMO systems. These methods originally emerged as a conjunction of linear algebra, geometry and system theory and compared to the classical identification methods [4], they provide user with several advantages such as numerical robustness, natural extension to MIMO systems, etc. There are, however, also some drawbacks, e.g. lack of satisfactory number of data samples, proper order of excitation or strong noise contamination can lead to poor identification results. Black-box identification, such as subspace identification methods, rely only on experimental data, that is, they may result in biased models [5].

Prior information can significantly improve identification results, however, current algorithms are not able to provide satisfactory results for MIMO systems. Previous works made use of Bayesian framework [5] but did not present method which would treat MIMO system in a satisfactory manner. This paper, in contrary, presents a new algorithm of incorporation prior information, which is built-in directly into system matrices B and D and does not make use of the covariance matrix. Proposed algorithm enables treating MIMO systems in a natural way using state-space approach.

The paper is organized as follows: Section II provides an insight into the building-up of the matrices used in subspace algorithms and formulates the general identification algorithm. Section III describes incorporation of prior information (PI) in subspace identification framework and shows two special cases of PI, knowledge of static gain and input-output feedthrough. Section IV presents identification results of previously described algorithms. The objective of the identification was creation of proper model (in sense of fit and controllability) of a real, eight-floor building. Future development is outlined in Section V and the paper is concluded with Section VI.

II. SUBSPACE IDENTIFICATION

A. Problem Statement

The objective of the subspace algorithm is to find a linear, time invariant, discrete time model in an innovative form

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + Ke(k) \\ y(k) &= Cx(k) + Du(k) + e(k), \end{aligned} \qquad (1)$$

based on given measurements of the input $u(k) \in \mathbb{R}^m$ and the output $y(k) \in \mathbb{R}^l$ generated by an unknown stochastic system of order n, which is equivalent to the well-known stochastic model

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + w(k) \\ y(k) &= Cx(k) + Du(k) + v(k), \end{aligned} \qquad (2)$$

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[1]Project OptiControl running at ETH Zurich aims at exploiting model predictive control strategies in control of the buildings (http://www.opticontrol.ethz.ch/). Similar project, ProjectEleven, runs at Czech Technical University in Prague (http://www.projecteleven.eu/).

with covariance matrices Q, S and R of process and measurement noise sequences as follows:

$$\mathrm{cov}(w,v) = \tag{3}$$
$$E\left(\begin{bmatrix} w(p) \\ v(p) \end{bmatrix}\begin{bmatrix} w^T(q) & v^T(q) \end{bmatrix}\right) = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}\delta_{pq} \geq 0,$$

and with A, B, C, and D denoting system matrices and K and e in (1) are Kalman gain – derived from the Algebraic Riccati Equation (ARE) [6], and white noise sequence, respectively.

Loosely speaking, the objective of the algorithm is to determine the system order n and to find the matrices A, B, C, D and K.

B. Matrices Used in Subspace Algorithm

Notation and building-up of the matrices as follows further on were adopted as in [7]. Upper index d denotes deterministic subsystem, while the upper index s denotes stochastic subsystem.

1) Data Matrices: Input block Hankel matrix is built-up from input data as follows:

$$U_{0|2i-1} = \left(\begin{array}{ccccc} u_0 & u_1 & u_2 & \cdots & u_{j-1} \\ u_1 & u_2 & u_3 & \cdots & u_j \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{i-1} & u_i & u_{i+1} & \cdots & u_{i+j-2} \\ \hline u_i & u_{i+1} & u_{i+2} & \cdots & u_{i+j-1} \\ u_{i+1} & u_{i+2} & u_{i+3} & \cdots & u_{i+j} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{2i-1} & u_{2i} & u_{2i+1} & \cdots & u_{2i+j-2} \end{array}\right), \tag{4}$$

which can be rewritten as

$$\left(\frac{U_{0|i-1}}{U_{i|2i-1}}\right) = \left(\frac{U_p}{U_f}\right) \tag{5}$$

with matrix U_p containing the past inputs and U_f containing the future inputs. The same logic holds for outputs $y(k)$ and noise $e(k)$. Change of indices in (5) results in

$$\left(\frac{U_{0|i}}{U_{i+1|2i-1}}\right) = \left(\frac{U_p^+}{U_f^-}\right). \tag{6}$$

Input and output Hankel matrices can be grouped as follows:

$$W_p = \left(\frac{U_p}{Y_p}\right) \quad , \quad W_p^+ = \left(\frac{U_p^+}{Y_p^+}\right). \tag{7}$$

2) System Related Matrices: Extended ($i > n$) observability (Γ_i) and reversed extended controllability (Δ_i) matrices for deterministic and stochastic subsystems, respectively are defined as follows:

$$\Gamma_i = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{i-1} \end{pmatrix} \tag{8}$$

$$\Delta_i^d = \begin{pmatrix} A^{i-1}B & A^{i-2}B & \dots & AB & B \end{pmatrix} \tag{9}$$

$$\Delta_i^s = \begin{pmatrix} A^{i-1}K & A^{i-2}K & \dots & AK & K \end{pmatrix} \tag{10}$$

The lower block triangular Toeplitz matrix for deterministic and stochastic subsystem, respectively are defined as

$$H_i^d = \begin{pmatrix} D & 0 & \dots & 0 \\ CB & D & \dots & 0 \\ CAB & CB & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{i-2}B & CA^{i-3}B & \dots & D \end{pmatrix}, \tag{11}$$

$$H_i^s = \begin{pmatrix} I & 0 & \dots & 0 \\ CK & I & \dots & 0 \\ CAK & CK & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{i-2}K & CA^{i-3}K & \dots & I \end{pmatrix}, \tag{12}$$

and Kalman state sequence as a sequence generated by a bank of non-steady state Kalman filters [8], working in parallel on each other of the columns of the matrix W_p.

C. General Algorithm

The entry point to the algorithm are input-output equations as follows:

$$\begin{aligned} Y_p &= \Gamma_i X_p + H_i^d U_p + H_i^s E_p \\ Y_f &= \Gamma_i X_f + H_i^d U_f + H_i^s E_f \\ X_f &= A^i X_p + \Delta_i^d U_p + \Delta_i^s E_p. \end{aligned} \tag{13}$$

Oblique projection as described in [9], [7] is the main tool used in subspace methods. It is defined as follows:

$$\mathcal{O}_i = Y_f \mathop{/}_{U_f} W_p, \tag{14}$$

or, equivalently,

$$\mathcal{O}_i = Y_f \begin{pmatrix} W_p^T & U_f^T \end{pmatrix} \begin{pmatrix} W_p W_p^T & W_p U_f^T \\ U_f W_p^T & U_f U_f^T \end{pmatrix}^{\dagger} \begin{pmatrix} I_{l\times l} \\ 0 \end{pmatrix} W_p, \tag{15}$$

where l is a number of outputs and $(\bullet)^{\dagger}$ is Moore-Penrose pseudoinverse. It has been shown ([7]), that

$$\mathcal{O}_i = \Gamma_i \widetilde{X}_i, \tag{16}$$

where $\widetilde{X}_i$ is Kalman filter state sequence. The order of the system can be determined from analysis of singular values obtained using singular value decomposition (SVD) of $W_1 \mathcal{O}_i W_2$, where W_i are weighting matrices of appropriate size and determine resulting state space basis as well as importance of particular element of $\mathcal{O}_i$. This decomposition also yields extended observability matrix Γ_i and Kalman filter states $\widetilde{X}_i$.

Algorithm continues from either Γ_i or $\widetilde{X}_i$ in a slightly different manner depending on particular subspace identification algorithm, however, both ways lead to a computation of system matrices A and C using least squares method.

Computation of system matrices B and D is the next step, such that matrices A and C acquired in previous step. Different approaches for matrices determination are addressed in detail in [7]. This step is crucial for the incorporation of the prior information and will be discussed in detail in the following section.

The algorithm concludes with computation of Kalman gain matrix K in a standard way using state and output noise covariance matrices (3) which are computed from residuals of the previous computations.

III. Incorporation of Prior Information

Prior information (PI) is a good tool for improvement of identification results. Its incorporation can be considered as a bridge between classical identification approaches ([4]) based on time response of unknown system on e.g. step or impulse response, and statistical based identification methods. System properties such as steady state gain, settling time, asymptotic stability, dominant time constants, smoothness of step response etc. can be used in classical approach to determine the unknown system. The question is, how to involve at least some of these properties into statistical based identification, and in particular, into 4SID methods.

Several methods dealing with above problem have been proposed. They can be generally classified into four groups.

1) Bayesian framework: This method can be characterized as a natural way for incorporation of PI because it allows inference of prior estimate of unknowns system parameters with information retrieved from measured data. Resulting posterior conditional probability function can be obtained using Bayesian rule ([10])

$$p(\theta|y) \propto l(\theta|y)p(\theta),$$

where $p(\theta)$ is prior probability density function of parameters and $l(\theta|y)$ the likelihood function for measured data.

Although many satisfactory results were proposed for incorporation of PI into ARX or ARMAX model identification ([10]), similar strategies do not work well for the class of 4SID methods. This problem is treated in [5], but favorable results are given only for multiple input single output (MISO) systems, because presented algorithm ([11]), based on structured weighted lower rank approximation (SWLRA), does not exist for MIMO systems.

2) Direct incorporation of system properties into 4SID algorithms: The following section tries to sketch out identification algorithm in simplified way. The incorporation of all conceivable kinds of PI is shown.

- Computation of extended observability matrix and state vector sequence

$$W_1 \mathcal{O}_i W_2 = \Gamma_i X_i.$$

  Different 4SID algorithms make use of different rules for computation of these matrices ([7]).
- Computation of system matrices A and C based on Γ_i using least squares method.
- Determination of matrices B and D and possible incorporation of prior information in this step. Solution will be addressed in Section III-A.
- Kalman gain computation.

3) Artificial data: Generation of data with desired properties is yet another approach how to deal with the weak point of 4SID, its black-box character (and associated statistical problems). Such data can contain trends that represent system in a decoupled form (connection of particular input to particular output etc.). As the ratio between artificial and measured data is unknown, the only way how to address this problem is trial and error method.

4) Frequency domain identification methods: Yet another approach for system identification is use of frequency domain methods. It was shown ([12]) that this approach leads to maximum likelihood formulation of the frequency domain estimation problem. Even though there were some proposals ([13]) how to incorporate prior information into identification algorithm, it is still an open problem and a topic of ongoing research.

In the following incorporation of prior information will be addressed:

A. Knowledge of Static Gain

Subspace identification process consists of several parts. Each of them corresponds to a particular property of resulting system. Matrix A contains dynamics of states, while matrix C transfers dynamics to the outputs. Therefore, the system input/output structure is influenced mainly by determination of matrices B and D, with A and C fixed. Hence, the key idea is to involve prior information about steady state gain into latter matrices.

Let matrices A and C have already been computed by some 4SID algorithm (e.g. [7]). Knowledge of these matrices can be exploited to compute such matrices B and D, that lead to desired steady state behavior. This is possible thanks to the fact, that the sum of elements of impulse response is equal to the steady state:

$$D + CB + CAB + CA^2B + \ldots = G, \tag{17}$$

$$\begin{pmatrix} I_{l\times l} & \sum_{k=0}^{\infty} CA^k \end{pmatrix} \begin{pmatrix} D \\ B \end{pmatrix} = G, \tag{18}$$

where G is a matrix of steady state gains (g_{ij} is a steady state gain from the j-th input to i-th output)

$$G = \begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1m} \\ g_{21} & g_{22} & \cdots & g_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ g_{lx1} & g_{l2} & \cdots & g_{lm} \end{pmatrix}. \tag{19}$$

In case of asymptotically stable matrix A, the following holds (Neumann series convergency theorem [14]):

$$(I_{n\times n} - A)^{-1} = \sum_{k=0}^{\infty} A^k. \tag{20}$$

Finally, we get resulting formula, which represents the additional set of constraints that have to be fulfilled:

$$\underbrace{\begin{pmatrix} I_{l\times l} & C(I_{n\times n} - A)^{-1} \end{pmatrix}}_{\Gamma_s} \begin{pmatrix} D \\ B \end{pmatrix} = G, \tag{21}$$

Consider any 4SID algorithm that computes matrices B and D after A and C being already computed. The computation is performed using least squares method as follows:

$$B, D = \arg\min_{B,D} \left\{ \left|\left| M - L \begin{pmatrix} D \\ B \end{pmatrix} \right|\right|_F \right\}, \tag{22}$$

where $||\bullet||_F$ denotes Frobenius norm, and M and L are appropriate size matrices defined in [7]. It must be said in this place, that these matrices are defined differently for each 4SID algorithm.

Incorporating constraints (21) can be done in two possible ways:

- Solve least squares problem with equality constraints

$$B, D = \arg\min_{B,D} \left\{ \left|\left| M - L \begin{pmatrix} D \\ B \end{pmatrix} \right|\right|_F : \Gamma_s \begin{pmatrix} D \\ B \end{pmatrix} = G \right\}. \tag{23}$$

- Solve weighted least squares problem

$$B, D = \arg\min_{B,D} \left\{ \left|\left| \begin{pmatrix} M \\ G \end{pmatrix} - \begin{pmatrix} L \\ \Gamma_s \end{pmatrix} \begin{pmatrix} D \\ B \end{pmatrix} \right|\right|_{F,W} \right\}, \tag{24}$$

  where W is user-defined weighting matrix that guarantees the desired steady state behavior. Particular input-output channel can be selected by appropriate choosing of W.

Computation of matrices B and D in some 4SID algorithms is based on vectorization and Kronecker product, that is:

$$B, D = \arg\min_{B,D} \left\{ \left|\left| \text{vec}\,\overline{M} - \overline{L}\ \text{vec} \begin{pmatrix} D \\ B \end{pmatrix} \right|\right|_F \right\}. \tag{25}$$

Using vectorization and Kronecker product, the set of equality constraints (21) can be expressed in following manner:

$$(I_{m\times m} \otimes \Gamma_s)\text{vec} \begin{pmatrix} D \\ B \end{pmatrix} = \text{vec}\, G\,, \tag{26}$$

which can be readily included in either (23) or (24).

B. Knowledge of input-output feedthrough

Oftentimes in industrial applications, the input-output feedthrough of the system to be identified is known in advance. In fact, it is not a rare phenomenon, that there is no input-output feedthrough present in the system, that is, the system matrix D is equal to zero. This will be treated in the following:

Consider again (22), the computation of matrices B and D, that is, the very last step of subspace identification algorithm as proposed in [7]. Matrix D can be forced to be zero by a computation of (22) or (25) using modified matrix L by the elimination of the columns corresponding to matrix D. The set of omitted columns differs for two distinct algorithms:

1) Without Kronecker product: Solution to this problem is given by omitting first l columns of matrix L in (22), where l corresponds to the number of outputs.

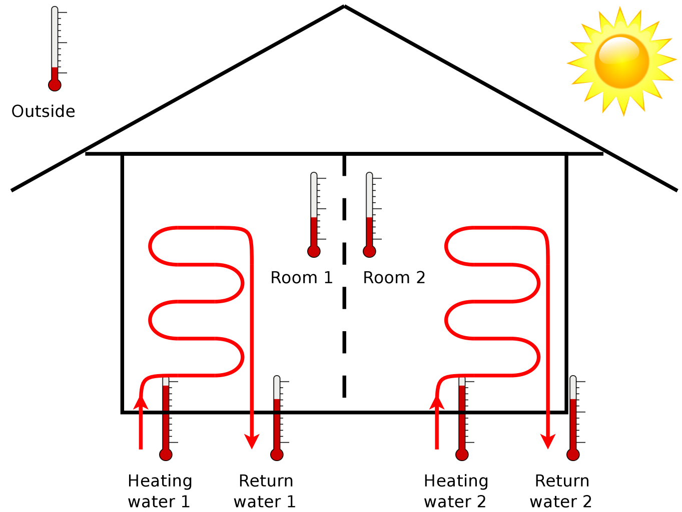

Fig. 1. Simplified scheme of model identification setup.

2) With Kronecker product: This situation is more complicated than in the first case due to vectorization and Kronecker product, nevertheless the selection of columns of L_1 is determined by indexes in set I, given as:

$$I = \{k(l+n) - n + 1, k(l+n) - n + 2, \dots k(l+n), \tag{27}$$

where $k = 1, 2, \dots m$.

IV. Identification Results

Proposed algorithms were implemented in Scilab[2] and then applied to data gathered from HVAC system of the building of the Czech Technical University in Prague. The simplified scheme of one building block consisting of three inputs (outside temperature, heating water 1, heating water 2) and four outputs (room temperature 1, return water 1, room temperature 2, return water 2) is depicted in Fig. 1.

Data from such an industrial environment do not always have sufficient quality, they suffer from strong noise contamination, occurrence of outliers, low excitation, etc. In our case, there is a strong multi-collinearity present in the data, that is, the conventional control strategies, which have been used for maintenance of desired temperature levels, drive both courses (north and south course, as well) of heating water, so that return waters and room temperatures had similar behavior and were strongly correlated. Black-box identification approach was not able to carry out this problem. Prior information about system structure i.e. steady state gain or/and no presence of input-output feedthrough had to be incorporated to get desired results. This can be seen in Fig. 2, where the step responses of models identified by different 4SID approaches are shown. Prior knowledge about

[2]Open source scientific software package for numerical computations (http://www.scilab.org/)

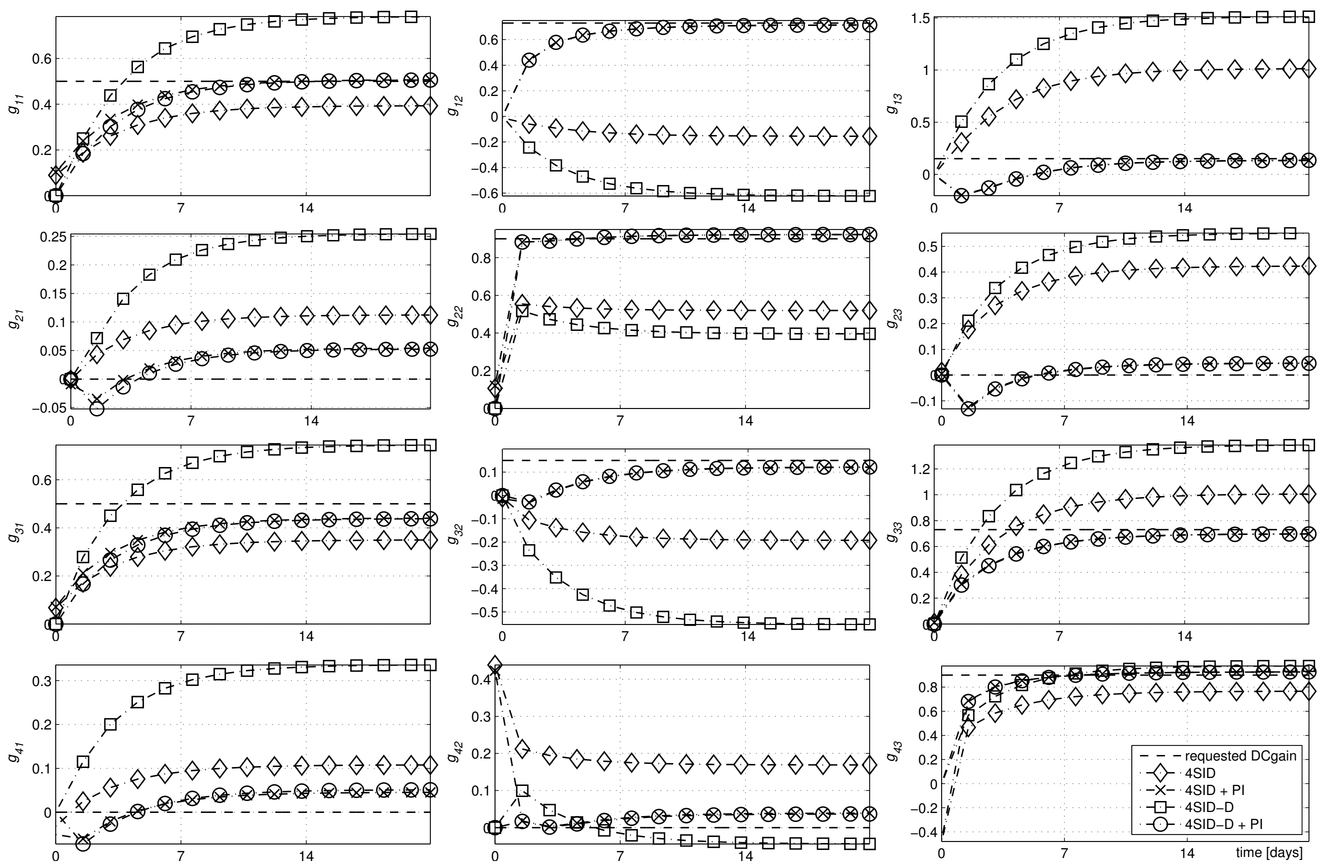

Fig. 2. Comparison of step responses of systems identified using different algorithms. There is significant improvement in identification results using prior information about steady state gain (dashed lines).

steady state gain was in this case selected as follows:

$$G = \begin{pmatrix} 0.5 & 0.75 & 0.15 \\ 0 & 0.9 & 0 \\ 0.5 & 0.15 & 0.75 \\ 0 & 0 & 0.9 \end{pmatrix}.$$

These 4SID methods come, in general, from robust combined deterministic and stochastic algorithm as introduced in [7]:

- 4SID – version without changes.
- 4SID+PI – Steady state gain was included using (26). Matrix D is not set to zero.
- 4SID-D – Matrix D is set to zero but steady state gain is not included.
- 4SID-D+PI – Both types of PI information, i.e. zero D and steady state gain are incorporated.

The same models were verified against validation data by open-loop simulation, see Fig. 3. Both figures prove the superiority of the identification algorithm with PI included. Identification results can be summed-up as follows:

- *Zero D matrix.* There is almost no difference in results between robust combined algorithm (full matrix D) and algorithm with zero D matrix. This is useful especially in cases, when nonzero matrix D has no physical meaning in many industrial applications.
- *Prior information in matrices B and D.* The incorporation of known static gain into identification algorithm has different consequences for deterministic and stochastic (in sense of system with noise) algorithms. In case of deterministic algorithm, the prior information is able to substitute the lack of information caused by noise (no presence of Kalman filter) and significantly improves identification results. In many cases, it is even not possible to identify system with noise using deterministic algorithm without knowledge of prior information due to the insufficient information and noise contamination and this can be rectified using prior information. In case of stochastic algorithm, the differences in fit between algorithm with and without prior information is not major, however, the incorporation of prior information enables creation of the model which has properties equivalent to real physical system and is valid for control.
- *Sensitivity of true value of PI.* The price for the better identification performance in case of PI incorporation must be paid by greater sensitivity to the changes in parameters, that is, even the slight change in parameters aggravates identification results (in sense of fit). The importance of prior information in respective parameters can be decided by weighting matrix in (24).

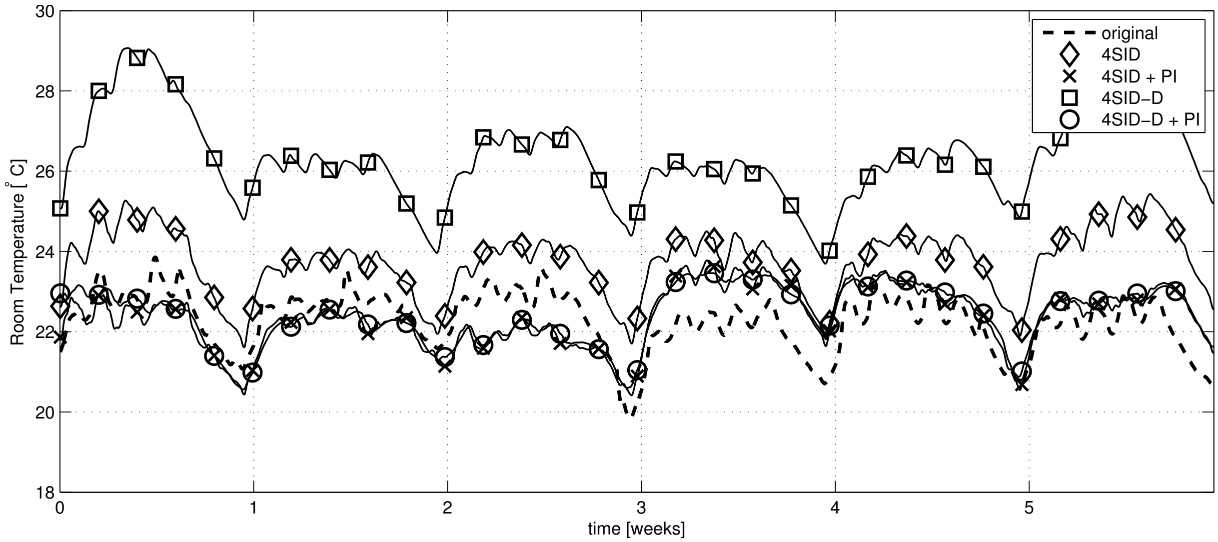

Fig. 3. Comparison of different identification strategies: open-loop simulation.

V. Future Development

As mentioned in Section III there is no properly working SWLRA algorithm for MIMO systems. This is still a topic of ongoing research. In case of successful solving of this problem, the prior information could be incorporated by means of Bayesian network as proposed by [5] even to MIMO systems. Yet another approach was presented in this article via direct incorporation PI into system matrices B and D. There is, however, PI of certain type (e.g. dynamics) which could be incorporated directly into matrices A or C, however, this approach is still unknown and topic of possible research as well.

VI. Conclusions

The proposed algorithm presents incorporation of PI into the subspace identification methods. The incorporation is performed directly into system matrices B and D, thus enables certain type of prior information, e.g. static gain. The incorporated PI is able to significantly improve identification results and substitute the lack of information in input-output data. Moreover, it notably improves model for control purposes by approaching to physical system structure. However, the quality of identification is sensitive to the accuracy of prior estimate of parameters. The constructed model has been used for temperature control in real operation of the 8-floor building of the Czech Technical University in Prague. The predictive control with model identified using algorithm proposed in this paper proved to save 23% of energy required by the weather-compensated controller.

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